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On Sequential Minimax*

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Under certain conditions, it is shown in this paper that a minimax sequential solution for a decision problem is a fixed sample size solution. This investigation is initiated by and applied to a specific problem, namely, the search for a maximum of a unimodal function. For this problem an ϵ-minimax sequential solution for a fixed sample size has already been given by J. Kiefer [1].

Consider the following search problem.

We have to estimate $x^{(f)}$ the place of the maximum of a function f by using an interval estimate $[s, t]$ in which $x^{(f)}$ lies. We know $f \in \mathscr{F}$ the class of all unimodal functions on the interval $[0, 1] \equiv I$ into the reals R. That means, for each $f \in \mathscr{F}$ there exists $x^{(f)} \in I$ such that,

f is strictly increasing for $x \leqslant x^{(f)}$ and strictly decreasing for $x > x^{(f)}$ or strictly increasing for $x < x^{(f)}$ and strictly decreasing for $x \geqslant x^{(f)}$. (1)

We are allowed observations on f at points of I before we make our estimate $[s, t]$; we restrict ourselves to estimating procedures that do not randomize the choice of places of observation x_i, and for which the interval estimate $[s, t]$ always contains the true maximum $x^{(f)}$. This will be assumed in the following whenever we talk of an estimating procedure.

The loss due to such an estimate is the length of our estimating interval, $t - s$.

We first consider the problem which was solved by Kiefer: we are allowed exactly n observations that can be taken sequentially and we want to minimize the maximum possible loss. Following Kiefer's notation we have:

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A sequential n-observation procedure is for $n \geqslant 2$

$$T_n = \{x_1, g_2, ..., g_n, \qquad s, t\} \tag{2}$$

where x_1 is the first place of observation, $x_1 \in I$, the other places of observations x_k are given by the functions $g_k : I^{k-2} \times R^{k-1} \to I$ $(2 \leqslant k \leqslant n)$ which are functions of the former x_i's and $f(x_i)$'s. The interval estimate given by $[s, t]$ depends on x_i and $f(x_i)$, s and t are functions on: $I^{n-1} \times R^n$ into I with $s \leqslant t$ and $[s, t]$ contains $x^{(f)}$ for every $f \in \mathscr{F}$. We denote the class of all such procedures by $\mathscr{T}_n$.

J. Kiefer [1] found for each given $n \geqslant 2$ and for every $\epsilon > 0$ a procedure $T_n^*(\epsilon) \in \mathscr{T}_n$ that is ϵ-minimax in $\mathscr{T}_n$; i.e., if we denote the loss from using T_n on f by $L(T_n, f)$ then:

$$\operatorname*{Sup}_{f \in \mathscr{F}} L(T_n^*, f) \leqslant \operatorname*{Sup}_{f \in \mathscr{F}} L(T_n, f) + \epsilon \qquad \text{for all} \quad T_n \in \mathscr{T}_n . \tag{3}$$

Denote by L_n^* Kiefer's lower bound for the maximum loss

$$\operatorname*{Sup}_{f \in \mathscr{F}} L(T_n^*, f) \leqslant L_n^* + \epsilon \leqslant \operatorname*{Sup}_{f \in \mathscr{F}} L(T_n, f) + \epsilon \qquad \text{for all} \quad T_n \in \mathscr{T}_n .$$

L_n^* is known to be $1/U_{n+1}$, U_n being the nth Fibonacci number defined as follows: $U_0 = 0$, $U_1 = 1$, $U_n = U_{n-2} + U_{n-1}$ for $n \geqslant 2$. Let us now change the problem. Instead of fixing the allowed number of observations, assume a cost $c > 0$ for each observation; we want to minimize the total cost $R = R(T, f)$, $R =$ (no. of observations taken) $\cdot\, c +$ (length of interval).

The important change is that now we allow a stopping rule that may decide at each stage whether to stop and take an estimate or to take more observations. Conceivably this would enable us to take advantage of "lucky" cases where we find very early that $x^{(f)}$ lies in a small given interval. However, as we keep the minimax criterion we suspect immediately that this generalization will not give us an advantage, since the minimax criterion judges a procedure precisely by the "unlucky" f's. Still we find that a few conditions have to be checked before this suspicion can be verified.

A sequential procedure T is

$$T = \{\delta_0, l_0, x_1, \delta_1, l_1, g_2, \delta_2, l_2, g_3, \delta_3, l_3, ...\} \tag{4}$$

where x_1 and g_k $(k \geqslant 2)$ are as in (2). δ_k, $(k = 0, 1, ...)$: $I^k \times R^k \to [0, 1]$ is the probability of stopping after k observations given the first k observations, (usually the δ_k's will be 0 or 1) for any given infinite sequence $x_1, f(x_1), x_2, f(x_2) \ldots \sum_{0=k}^{\infty} \delta_k(T, \text{seq.}) = 1$ but each δ_k depends only on the first $2k$ elements of the sequence. As each sequence is completely determined by T and f we can also write $\delta_k(T,f)$ and $\sum_{0=k}^{\infty}\delta_k(T,f) = 1 \;\forall\, T \in \mathscr{T}$ and $f \in \mathscr{F}$. l_k is the interval estimate if we stop after k observations, equivalent to the two functions s, t of (2) and here also we require $x^{(f)} \in l_k$ for all f and all k.

Let $\mathscr{T}$ be the set of all such sequential procedures T. We shall show that for a certain $n_0 = n_0(c)$, $T_{n_0}^*$ is ϵ minimax among $\mathscr{T}$. But first let us state this a little more generally.

THEOREM. *Let $\mathscr{F} = \{f\}$ be a set of "states of nature," X be a set of possible observations on f, and $\{d_i\} \equiv D_i(x_1, ..., x_i, f(x_1), ..., f(x_i))$ be a set of admissible decisions, given the first i observations. Assume a loss function $L(d, f)$, $0 \leqslant L(d, f) \leqslant 1$.*

Fixed sample size decision procedures T_n, or general sequential decision procedures T, will be defined in analogy with (2) and (4) as d_n and d_i replace $[s, t]$ and l_i.

If *for every number n (and $\epsilon > 0$) there exists a procedure $T_n{}^* \in \mathscr{T}_n$ and a number $L_n{}^*$ such that*

$$\operatorname*{Sup}_{f\in\mathscr{F}} L(d_n(T_n{}^*, f), f) + (-\epsilon) \leqslant L_n{}^* \leqslant \operatorname*{Sup}_{f\in\mathscr{F}} L(d_n(T_n, f), f) \qquad \forall\, T_n \in \mathscr{T}_n .$$

If also

(A) *the sequence $L_n^* - L_{n-1}^* \downarrow 0$ is strictly decreasing and*

(B) *for each general procedure T and any given integer $k > 0$, there exists $f^* = f^*(k, T) \in \mathscr{F}$ such that*

$$L(d_i(T, f^*), f^*) \geqslant L_i{}^* \qquad \text{for} \quad i \leqslant k.$$

Then *if we define n_0 such that*

$$L_{n_0}^* - L_{n_0-1}^* > c \geqslant L_{n_0+1}^* - L_{n_0}^* \tag{5}$$

or $n_0 = 0$ if (5) does not hold for any n, $T_{n_0}^*$ is (ϵ) minimax among all non-randomized sequential procedures $T \in \mathscr{T}$.

That means (R being again the total cost)

$$\operatorname*{Sup}_{f\in\mathscr{F}} R(T_{n_0}^*, f) = \operatorname*{Sup}_{f\in\mathscr{F}} L(d_n(T_{n_0}^*, f), f) + n_0 c$$

$$\leqslant \operatorname*{Sup}_{f\in\mathscr{T}} \sum_{i=0}^{\infty} [L(d_i(T, f), f) + ic]\delta_i(T, f) + (\epsilon) \tag{6}$$

$$\text{for all} \quad T \in \mathscr{T}.$$

PROOF: Let the integer $k > 0$ be such that

$$kc \geqslant 1. \tag{7}$$

If T is any procedure in $\mathscr{T}$, let $f^* = f^*(k, T)$ fulfill condition (B). Then:

$$R(T,f^*) = \sum_{i=0}^{\infty}[L(d_i(T,f^*) + ic]\delta_i(T,f^*)$$

$$\geqslant [\sum_{i=0}^{k-1}[L(d_i(T,f^*),f^*) + ic]\delta_i(T,f^*) + k \cdot c\sum_{i=k}^{\infty}\delta_i(T,f^*)$$

$$\geqslant \sum_{i=0}^{k-1}(L_i{}^* + ic)\delta_i(T,f^*) + kc\sum_{i=k}^{\infty}\delta_i(T,f^*) \quad \text{by assumption } (B).$$

From (A) and (5) and as $L \leqslant 1$ it follows that

$$kc \geqslant 1 \geqslant L_0{}^* \geqslant L_n{}^* + n_0c \tag{8}$$

as $L_0{}^* - L_{n_0} > n_0c$, also $L_i{}^* + ic \geqslant L^*_{n_0} + n_0c$ for $i = 1, 2, \ldots$. So

$$R(T,f^*) \geqslant [L^*_{n_0} + n_0c] \sum_{i=0}^{\infty}\delta_i(T,f^*) = L^*_{n_0} + n_0c \geqslant \operatorname*{Sup}_{f\in\mathscr{F}} R(T^*_{n_0},f) + (-\epsilon)$$

Q.E.D.

We have now to prove conditions (A) and (B) for our original problem.

Condition (A) holds in this case only for $n \geqslant 2$, as $L_0{}^* = L_1{}^*$, that will require choosing n_0 a little more carefully. To prove (A) for $n \geqslant 2$ we have to show only:

$$L^*_{n-1} - L_n{}^* > L_n{}^* - L^*_{n+1}, \tag{9}$$

which becomes for this problem

$$\frac{1}{U_n} - \frac{1}{U_{n+1}} > \frac{1}{U_{n+1}} - \frac{1}{U_{n+2}}$$

or

$$\frac{U_{n+1}}{U_n} - 1 > 1 - \frac{U_{n+1}}{U_{n+2}}$$

or

$$\frac{U_n + U_{n-1}}{U_n} + \frac{U_{n+1}}{U_{n+2}} > 2$$

or

$$\frac{U_{n-1}}{U_n} + \frac{U_{n+1}}{U_{n+2}} > 1. \tag{10}$$

We have

$$\frac{U_{n-1}}{U_{n-1}+U_{n-2}} \geqslant \frac{1}{2}$$

as $U_{n-1} \geqslant U_{n-2}$. The only time equality holds is for $n = 3$, so in all cases one of the two terms is $>\frac{1}{2}$ the other $\geqslant\frac{1}{2}$, establishing (10) and thus (9).

In choosing n_0 we first compare

$$2c \gtreqless L_0{}^* - L_2{}^* = 1 - \frac{1}{3}.$$

If $\geqslant$, take no observations.

If $<$, define n_0 by (5).

(B) After we have taken any number i of observations under any procedure T, we have an interval of uncertainty where $x^{(f)}$ must be but no subinterval of it is sure to contain $x^{(f)}$, with the information gained until now.

This interval, denoted by $V_i = V_i(T, f)$, is given by

$$V_i = [\max(0, x_j \mid j \leqslant i, x_j < x_m), \min(1, x_j \mid j \leqslant i, x_j > x_m)],$$

x_m being the place of the largest observation until now. (If $\max f(x_j)$ is obtained at 2 points $x_n < x_m$, $V_i = [x_n, x_m]$.)

Let us use V_i ambigously to denote both the interval and its length, which will be helpful and will not confuse us in the following.

As we required each procedure to give us an interval estimate that contains the true $x^{(f)}$, this interval must contain V_i at each stage i. That means the interval estimates $l_i(T, f)$, defined for a stop after i observations, must satisfy

$$_i(T,f) \supset V_i(T,f)$$

and

$$L(l_i(T,f),f) \geqslant V_i(T,f).$$

So if we prove that condition (B) holds for all T with the V_i's replacing the l_i's it will certainly be true for the l_i's too.

So we want to prove the existence of $f^* = f^*(k, T)$ such that

$$V_i(T, f^*) \geqslant L_i{}^* \qquad \text{for} \quad i \leqslant k.$$

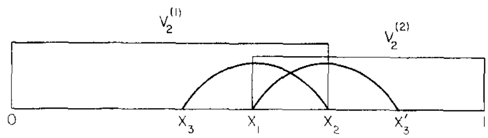
For any procedure T, $V_0 = V_1 = [0, 1]$. T observes x_1 and finding $f(x_1) = a$ it observes x_2, say $x_1 \leqslant x_2$ (without loss of generality) we have 2 possible V_2's, say $V_2^{(1)} = [0, x_2]$, $V_2^{(2)} = [x_1, 1]$, assume w.l.o.g. $V_2^{(1)} \geqslant V_2^{(2)}$. It is obvious that there exists f_1 and $f_2 \in \mathscr{F}$ that will give under T, $V_2(T, f_1) = V_2^{(1)}$ and $V_2(T, f_2) = V_2^{(2)}$ (we need only $f_i(x_1) = a$, $f_1(x_2) < a$, $f_2(x_2) > a$).

From $T_2{}^*$ being ϵ minimax we have $V_2^{(1)} \geqslant L_2{}^*$.

Given that f and T lead us to $V_2^{(1)}$ and then to observe x_3 (we may assume $x_3 \in V_2^{(1)}$ as otherwise $V_3 = V_2$), we again get two possible V_3 intervals $V_3^{(1)}$ and $V_3^{(2)}$ which are obtained from $V_2^{(1)}$ through the observations x_1 and x_3 in the same way as $V_2^{(1)}$ and $V_2^{(2)}$ were obtained from I by x_1 and x_2 and w.l.o.g. $V_3^{(1)} \geqslant V_3^{(2)}$. Again it is clear that there are functions in $\mathscr{F}$ for which T will lead to $V_3^{(1)}$ and others for which it will lead to $V_3^{(2)}$.

We claim again $V_3^{(1)} \geqslant L_3{}^*$.

Proof. Assume $V_3^{(1)} < L_3{}^*$, then let T_3^{**} be the three observation procedure that takes observation x_1, x_2 and if $V_2^{(1)}$ is obtained it takes x_3, while if $V_2^{(2)}$ is obtained it takes symmetrically $x_3{}^1 = x_1 + (x_2 - x_3)$; if $x_3{}^1 \notin V_2^{(2)}$ we can ignore it. This divides $V_2^{(2)}$ into two possible



intervals, $V_3^{(3)}$ and $V_3^{(4)}$. Because of symmetry and of $V_2^{(2)}$ being $\leqslant V_2^{(1)}$ we have $V_3^{(3)}$ and $V_3^{(4)} \leqslant V_3^{(1)} < L_3{}^*$. We get a contradiction to the fact that $T_3{}^*$ is ϵ-minimax.

So we can go on and always divide the larger $V_j^{(i)}$ into two possible V_{j+1}'s, $V_{j+1}^{(1)} \geqslant V_{j+1}^{(2)}$ (w.l.o.g.) until we obtain after k observations

$$V_k^{(1)} \subset V_{k-1}^{(1)} \subset --- \subset V_2^{(1)},$$

claim $V_j^{(1)} \geqslant L_j{}^*$, $j \leqslant k$.

Proof: Assume $V_j^{(1)} < L_j{}^*$ for some $j \leqslant k$. Then let T_j^{**} be defined in analogy with $T_3^{**} \cdot T_j^{**}$ takes the same observations that T took leading to $V_j^{(1)}$ and $V_j^{(2)}$, an observation symmetric to x_j in $V_{j-1}^{(1)}$ is taken in $V_{j-1}^{(2)}$ and then the symmetry is carried back to $V_{j-2}^{(2)}$ up to $V_2^{(2)}$ on which all observations x_3 up to x_j are chosen symmetrically to those on $V_2^{(1)}$. Each $V_i^{(2)}$ is smaller than the corresponding $V_i^{(1)}$. This, along with the symmetry of T_i^{**}, leads to

$$V_j(T_j{}^{**}, f) \leqslant V_j^{(1)} \qquad \text{for all} \quad f \in \mathscr{F}.$$

So by our assumption we obtain $V_j(T_j{}^{**}, f) < L_j{}^*$ for all $f \in \mathscr{F}$, a contradiction to the fact that $T_j{}^*$ is ϵ-minimax.

As $\exists f^* \in \mathscr{F}$ such that $V_j\,(T, f^*) = V_j^{(1)}$ for $j \leqslant k$ (which can be trivially constructed), we have completed the proof of condition (B) and shown that the theorem applies to our search problem.

The theorem can be applied immediately to the simpler search problem of finding a root (zero) of a monotone function on an interval I. Here with the same loss function we obtain a minimax solution by the Bolzano method of taking the next observation at the middle of the interval of uncertainty.

If the length of I is 1, we take

$$n_0 \quad \text{such that} \quad \frac{1}{2^{n_0+1}} \leqslant c < \frac{1}{2^{n_0}},$$

or in other words go on as long as one further observation can reduce the interval of uncertainty by an amount larger than c.

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